PhD proposal

Orthotropic maps for mesh generation

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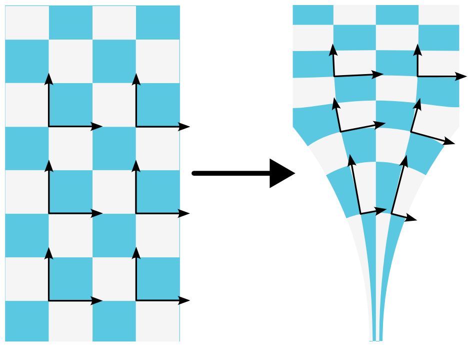
I. Context

Computing planar maps is a very old problem that has occupied geographers and mathematicians for centuries in order to best describe the earth at different scales. As a curved surface is stretched and sheared into the plane, the flat representation of countries or continents are no longer faithful to their original shapes. In this sense, there are no perfect maps, and any maps introduce a certain amount of distortion.

Idea. We propose to study the set of mappings which are entirely free from shear. Such maps only allow independent stretching in two (or three) orthogonal directions. Inspired by materials science, we refer to them as *"orthotropic"* maps. Intuitively, on a planar domain, we define at each point a reference frame aligned with the global refence system and locally attached to the material. Our degrees of freedom are the rotation of these frames and the independent scaling of the two vectors. This deformation of the domain transforms infinitesimal squares, initially aligned with the reference frames, into infinitesimal rectangles. This property is essential in applications such as mesh generation as discussed in Section III.



Challenges. The goal of this thesis is to develop a framework to study and to numerically compute orthotropic mappings with a particular focus on volumetric maps and atlases.

II. Methodology

In order to theoretically study orthotropic deformations, we use Cartan's method of moving frames [1]. The main idea is to define a system of orthogonal frames and scale functions that smoothly evolve in domain of arbitrary dimension and to characterize when they infinitesimally define a valid deformation. More precisely, let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a planar orthotropic map, then its Jacobian must be such that $\nabla f = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix} E^\top$, where $E: \mathbb{R}^2 \to SO(2)$ is a rotation field and $a, b: \mathbb{R}^2 \to \mathbb{R}$ are the scaling functions. The map is then locally characterized by the necessary condition: $\nabla \times \nabla f = 0$. We can rewrite this equation to fully express the infinitesimal rotation, which relates two infinitesimal close rotations E in terms of a, b and E. This necessary (and sufficient in many useful cases) integrability condition is linear with respect to a and b.

To make this theoretical analysis useful for applications, the student will work on the following tasks:

Task 1: Orthotropic parametrizations of surfaces. The first step is to explore the theoretical foundations of orthotropic deformations for surfaces. The goal is to discretize and efficiently solve the non-linear integrability equation.

Task 2: Orthotropic mapping in volumes. The integrability condition for the existence of an orthotropic map can be easily extended to the volumetric case. However, this additional dimension introduces new challenges. 3D rotations are no longer commutative making the optimization significantly more difficult, and as a result, our solver for surface maps may become obsolete.

Task 3: Beyond orthotropy. Discover which maps can be computed while keeping two key properties: a characterization by a small number of meaningful parameters and convergence under refinement to an injective map.

III. Orthotropic map applications for remeshing

In a modern version of the mapping problem, surfaces are represented by triangle meshes and the flat representation is not only used for information storage (texture, normal). In particular, by computing a cleverly constrained map to the plane and by overlying a regular grid in parameter space, we obtain a decomposition of the original surface into quadrangles (see Figure 1). This transformation of triangle mesh to a quadrangle mesh proves to be quite challenging but very useful in practice.

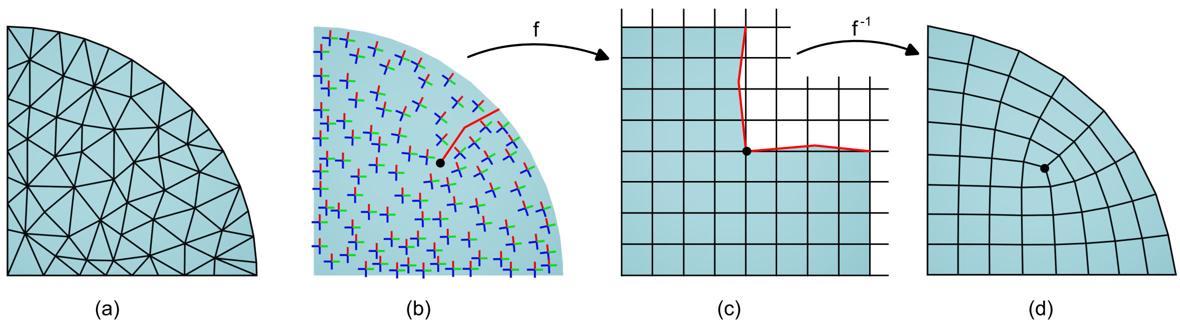
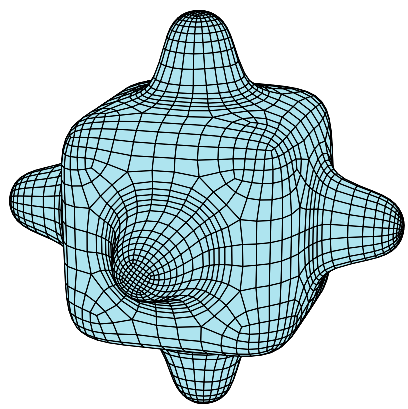
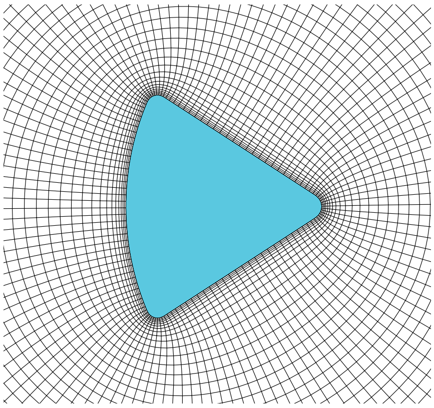


Figure 1: The quad meshing algorithm. Given a triangle mesh (a), a frame field is computed (b) defining the singularity position (black dot). The mesh is cut along the frame discontinuities (red line) and mapped to a grid aligned domain (c). The inverse deformation is applied to produce a quad mesh (d). This project aims at computing the map f and the singularity positions directly with the help of orthotropic maps.

Anisotropic remeshing. The objective of quad (or hex) remeshing is to generate a mesh that accurately approximate a target geometry while maintaining a fixed number of elements. When approximating a surface using a quad mesh, theoretical findings indicate that the edges should align with the principal (orthogonal) curvature directions, and the aspect ratio of the elements should be in proportion to the ratio of the principal curvatures [2] as in the inset figure. This result is a quite intuitive because regions with high curvature demand smaller elements for a precise approximation. Similarly, to reduce numerical errors in numerical simulations, the mesh should be denser in areas where the expected solution exhibits significant variations and less dense in areas where the solution is nearly flat. The theory suggests that the most accurate quad (or hex) mesh must have edges aligned with the (orthogonal) eigenvectors of the function's Hessian [3]. Clearly, both aspects of the approximation problem can be addressed by meshes with rectangular (or rectangular cuboid) elements, which can be extracted from an orthotropic map.



Numerical simulation. It is well-known that the finite element method exhibits improved convergence properties when elements approach perfect squares or cubes, and the convergence is not guaranteed in the presence of non-convex elements [4]. By enforcing rectangular elements through orthotropic mappings, we not only avoid non-convexity but also fulfill the requirements for the optimal convergence of the popular finite elements [5].



Our colleagues at CEA conducts numerous numerical simulations using quad and hex meshes, with a particular focus on boundary layer simulations that necessitate highly anisotropic elements to capture extreme physical phenomena in directions normal to the boundary (for example, Apollo 11 entering the atmosphere). In their current workflow, users manually remesh their models according to their specific needs, which is a time-consuming process, often taking weeks for an engineer to obtain the desired mesh. The CEA is actively engaged in research to reduce the user input during mesh generation. Orthotropic maps present a promising solution to address some of these challenges.

IV. Application and starting date

The PhD position starts at the latest in autumn 2025. The candidate must hold a master in computer science or in applied mathematics. Typically, a candidate with knowledge in differential geometry or/and finite element method is appreciated. This PhD offers the opportunity to visit (and work with) Franck Ledoux from CEA and Keenan Crane from Carnegie Mellon University. Applications can be sent in either French or English. To apply for the position, please send a CV to etienne.corman@cnrs.fr.

V. References

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